# LEXICOGRAPHICAL ORDER, RANGE OF INTEGRALS AND "BANG-BANG" PRINCIPLE

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INTRODUCTION. Let $J$ denote a compact interval, say $[0,1]$, $E$ -- an Euclidean n-space, $M$ -- the space of Lebesgue measurable functions of $I$ into $E$. For any $u, v \in M$ the equality $u = v$ will mean $u(t) = v(t)$ almost everywhere (a.e.) in $J$. The topology in $M$ will be that given by the convergence in measure.

The purpose of this paper is to study in detail the range of integrals of a subset $K \subset M$ which satisfies the following three conditions

(i) $K$ is closed in $M$ with respect to convergence in measure

(ii) $|\int_J u(\tau)d\tau| \leq m$ for each $u \in K$

(iii) If $u,v \in K$, $0 < t_1 < 1$, and $w(t) = u(t)$ if $0 \leq t < t_1$ and $v(t)$ if $t_1 \leq t \leq 1$, then $w \in K$.

The motivation to study the range of integrals of such a class $K$ comes from linear control theory. Indeed let us consider the system of the form

$$\dot{x}(t) = A(t)x(t) + f(t,u(t)), \tag{S}$$

where the function $f$ satisfies the well know Caratheodory

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conditions. Take as admissible control functions the class of Lebesgue measurable $u: I \to U$, where $U$ is a compact subset of an m-dimensional space. Any solution of (S) can be represented in the form $x(t) = X(t)(x_0 + \int_0^t v(\tau)d\tau)$, where $X(t)$ is the fundamental matrix solution of the corresponding homogeneous system, $x_0$ is the initial value for $t = 0$ and $v(t) = X^{-1}(t)f(t,u(t))$. It is easy to verify that the class

$$L = \{v: v(t) = X^{-1}(t)f(t,u(t)), u \text{ -- admissible}\}$$

satisfies conditions (i),(ii), and (iii). A basic result for the existence of a time-optimal solution for (S) is that the so-called attainable set

$$\Omega(t) = \{x: X(t)(x_0 + \int_0^t v(\tau)d\tau),\ v \in L\}$$

is convex, compact and continuous in $t$. Up to a linear transformation and a translation this set is seen to be the range of integrals over $L$. This result among others will be proved here but probably more interesting is an extension of LaSalle's "bang-bang" principle. Roughly speaking the "bang-bang" principle as stated by LaSalle (2) says that in general one can restrict the range $U$ of admissible controls to a subset $U_0$ without restricting the attainable set. In LaSalle's case $f$ was linear in $u$, $U$ was a compact cube and he proved $U_0$ to be the set of vertices of $U$. Later this result has been extended by several authors, cf. for example (1),(3),(4). Our extension of the "bang-bang" principle is Theorem I and states that there is a smallest subclass $K_0$ of $K$ satisfying (iii) but not necessarily (i) such that the range of integrals over $K_0$ is the same as over $K$. In LaSalle's case the restricted class of "bang-bang" controls satisfies (i), too.

The results presented here generalize those recently published by the author in (5). In (5) the class $K$ was given by $\{v \in M: v(t) \in G(t)\}$ where $G$ is a measurable map [cf. (6)] of $T$ into the space of compact subsets of $E$. In the situation concerning system (S) discussed above the set-valued map is given by $\{X^{-1}(t)f(t,u): u \in U\}$.

There is a close connection between our results and the Liapunov theorem on the range of non-atomic vector valued measures. For details we refer the reader to (5).

The following notations will be used. By $(x,y)$, $x$, $y \in E$, we denote the scalar product of $x$ and $y$, by $|x|$ the Euclidean norm of $x \in E$. Thus $|u|$ and $(u,v)$ if $u$, $v \in M$ will stand for the function taking $t \to |u(t)|$ and

and $t \to (u(t), v(t))$, respectively. By $I$ and $I_t$ we denote the integral operator $\int_J$ and $\int_0^t$ respectively. Thus $I(u) = \int_J u(\tau)d\tau$ and $I(K) = \{I(u): u \in K\}$.

LEXICOGRAPHICAL ORDER IN $E$ AND IN $M$. Let $x, y \in E$ and let $\{x_i\}$, $\{y_i\}$ denote the coordinates of $x$ and $y$ respectively with respect to a fixed coordinate system in $E$. We will write

$$x \leqq y \text{ iff } x_i = y_i \text{ for } i = 1, \ldots, k \text{ and if } k < n \text{ then } x_{k+1} < y_{k+1}. \quad (1)$$

In particular, $k$ may be equal 0. The relation (1) is the so-called lexicographical order in $E$ and it is easy to see that it is a linear order. If $n = 1$, then (1) is the natural order for reals. If, in (1), $k < n$ then we will use "$<$" instead of "$\leqq$". Since the order is linear, any finite subsets of $E$ admits a unique maximum with respect to (1). Thus we have

$$\underset{1 \leq i \leq s}{\text{lex.max}} \{x^i\} = x^j \text{ iff } x^i \leqq x^j \text{ for } i = 1, \ldots, s \quad (2)$$

If, $u, v \in M$ then we will write

$$u \leqq v \text{ iff } u(t) \leqq v(t) \text{ a.e. in } J \quad (3)$$

and refer to (3) as the lexicographical order in $M$. The order "$\leqq$" in $M$ is no longer linear but is a lattice, since for any finite set $\{u^i\}$, $1 \leqq i \leqq s$ of $M$ the lex. sup exists and we have

$$v = \underset{1 \leq i \leq s}{\text{lex sup}} \{u^i\} \text{ iff } v(t) = \underset{1 \leq i \leq s}{\text{lex.max}} \{u^i(t)\}. \quad (4)$$

We note the following obvious propositions.

<u>Proposition 1. If $u, v \in M$ are integrable and $u \leqq v$ then $I(u) \leqq I(v)$.</u>

<u>Proposition 2.</u> If $u \leqq v$ and $I(u) = I(v)$ then $u = v$.

<u>Proposition 3.</u> If $u, v \in M$ are integrable, $w = \text{lex.sup}\{u, v\}$, $I(u) = p = (p_i)$, $I(v) = q = (q_i)$, $I(w) = r = (r_i)$, $i = 1, \ldots, n$, and if $r_i = q_i = p_i$ for $i = 1, \ldots, k \leqq n$, then $u_i = v_i$ for $i = 1, \ldots, k$, where $u_i(t)$, $v_i(t)$ are coordinates of $u(t)$ and $v(t)$ respectively.

Notice that the lexicographical order in $E$ or $M$ depends on the coordinate system in $E$. Thus if $\xi = (x^1, \ldots, x^n)$,

$x^i \in E$, is a basis in $E$ then by "$\leq_\xi$" we will denote the lexicographical order corresponding to $\xi$. In the sequel we restrict ourselves to the orthonormal bases in $E$. Thus we will be interested in the set

$$\Xi = \{\xi : \xi = (x^1, \ldots, x^n), (x^i, x^j) = \delta_{ij}, i, j = 1, \ldots, n\};$$

where $\delta_{ij} = 1$ if $i=j$ and $0$ otherwise.

Let $A \subset E$ be compact, then to each $\xi \in \Xi$ there is a unique point denoted by $e(A,\xi)$ of $A$, which is the lexicographical maximum of $A$ with respect to "$\leq_\xi$", and is determined by the conditions: $e(A,\xi) \in A$ and $x \leq_\xi e(A,\xi)$ for each $x \in A$. The next proposition can be found in (7) in a slightly different form but for completeness we include here a detailed proof.

Proposition 4. Let $A \subset E$ be compact, then the set

$$B = \bigcap_{\xi \in \Xi} \{x : x \leq_\xi e(A,\xi)\} \tag{5}$$

is the convex hull of $A$. Moreover, the set

$$D = \{e(A,\xi) : \xi \in \Xi\} \tag{6}$$

is the profile of $\ddot{B}$ of $B$; that is, the set of extreme points of $B$.

Proof. Let $C \subset E$ be convex and let $p \notin C$. Then there is a $\xi \in \Xi$ such that

$$x <_\xi p \quad \text{for each} \quad x \in C. \tag{7}$$

If $n = 1$ then (7) is obvious. For $n$ arbitrary there is an $a \in E$, $|a| = 1$ such that $(p,a) \geq (x,a)$ for each $x \in C$. If $(p,a) > (x,a)$ for each $x \in C$, then (7) holds for any $\xi = (x^1, \ldots, x^n) \in \Xi$ if $x^1 = a$. If $(p,a) = (x,a)$ for some $x \in C$ then the set $C_1 = C \cap \{x : (x,a) = (p,a)\}$ is non-empty, convex and of dimension n-1 at the most, and $p$ does not belong to $C_1$ but does belong to the hyperplane containing $C_1$. Thus we have the same situation but in a smaller dimension. Therefore an easy induction argument completes the proof of (7). Let $C$ be now the convex hull of $D$ given by (6). It follows from (7) that if $p \notin C$ then $p \notin B$ given by (5). Hence $B \subset C$. But $B$ is convex and $D \subset A \subset B$. Therefore $C$ as the convex hull of $D$ is contained in $B$. Hence $C = B$ and $B$ given by (5) is the convex hull of $E$ and since $D \subset A \subset B$, $B$ is the convex hull of $A$ as well. In particular $B$ is compact. To end the proof, let us recall that a point $b \in B$ is an extreme point of $B$ if and only if $B\setminus\{b\}$ is convex. Let $b = e(A,\xi) \in D$. By (5), $B\setminus\{b\} = B \cap \{x : x <_\xi e(A,\xi)\}$. Manifestly the latter set is convex for each $\xi \in \Xi$ and we conclude that $D \subset \ddot{B}$. Suppose now that $b \in \ddot{B}$. Then $B\setminus\{b\}$ is convex and by (7) there is a $\xi \in \Xi$ such that $x <_\xi b$ for each $x \in B\setminus\{b\}$. Hence $b = e(B,\xi)$. It is easy to see by (5) that $e(A,\xi) = e(B,\xi)$ for each $\xi \in \Xi$. Therefore $b \in D$ and in consequence $\ddot{B} \subset D$ which completes the proof.

PRELIMINARY LEMMAS. In this section we will always assume that the class $K$ satisfies conditions (i), (ii) and (iii). A coordinate system in $E$ is fixed.

LEMMA 1. Let $\{A_i\}$, $1 \leq i \leq k$ be a decomposition of $J$ into $k$ disjoint measurable subsets. Let $\{u^i\}_{1\leq i\leq k} \subset K$. Put $u(t) = u^i(t)$ if $t \in A_i$. Then $u \in K$.

Proof. Since $A_i$ can be approximated arbitrarly closely by disjoint unions of intervals, therefore by (iii) $u$ can be approximated by a sequence $\{u^i\} \subset K$ converging to $u$ in measure. Thus (i) completes the proof.

LEMMA 2. The lexicographical order on $K$ is a lattice; that is if $u^i \in K$ for $i=1,\ldots,k$ then so does $v = \operatorname{lex\,sup}_{1\leq i\leq k} \{u^i\}$

Proof. By (4), $v(t) = u^i(t)$ if $t \in A_i\{t : u^i(t) = \operatorname{lex\,max}_{1\leq i\leq k}\{u^i(t)\}$, $u^j(t) < u^i(t)$ if $j < i\}$. It is easy to see that these $A_i$ satisfy the assumptions of Lemma 1. Hence the latter finishes the proof.

LEMMA 3. Let $u^i = (u_1^i,\ldots,u_n^i) \in K$ for $i=1,2,\ldots$ . Assume that $u_j^i \to u_j^o$ a.e. in $J$ if $j=1,\ldots,k-1$, $1\leq k\leq n$ and put $u_k^o = \limsup_i u_k^i$. Then there is a $v = (v_1,\ldots,v_n) \in K$ such that $v_j = u_j^o$ if $j=1,\ldots,k$.

Proof. Take an $\varepsilon > 0$. There exist $i_o$, sets $F,G \subset J$, $\mu(F) < \varepsilon$, $\mu(G) < \varepsilon$ and an integer $p$ such that

$$|u_j^i(t) - u_j^o(t)| < \varepsilon \quad \text{if} \quad 1\leq j\leq k-1,\ i \geq i_o \text{ and } t \in J\setminus F \tag{8}$$

and

$$\min_{i_o\leq i\leq i_o+p} |u_k^i(t) - u_k^o(t)| < \varepsilon \quad \text{if} \quad t \in J\setminus G. \tag{9}$$

Put $A_s = \{t : |u^{i_o+s}(t) - u_k^o(t)| < \varepsilon$ and $|u_k^{i_o+r}(t) - u_k^o(t)| \geq \varepsilon$ for $r < s\}$, $s = 0,1,\ldots,p$. Clearly the $A_s$ are measurable and disjoint and, by (9), $\bigcup_{s=0}^{T} A_s \supset J\setminus G$. Define $v(t) = u^{i_0+s}(t)$ if $t \in A_s$, $s = 0,1,\ldots,p$ and $v(t) = u(t)$ if $t \in J\setminus \bigcup_{=0}^{T} A_s$, where $u \in K$. By Lemma 1, $v \in K$ and by (8) and (9) we get

$$|v_j(t)-u_j^o(t)| < \varepsilon \quad \text{if} \quad t \in J\setminus(F\cup G),\ 1 \leq j \leq k. \tag{10}$$

Ineq. (10) shows that a sequence $v^i \in K$ can be defined such that $v_j^i \to u_j^o$ a.e. in J for $j=1,\ldots,k$. If $k=n$ then the last statement and (i) proves Lemma 3. If $k < n$, then it proves that $\{v^i\}$ satisfies assumptions of Lemma 3 for k increased by 1. Hence the proof can be completed by induction.

COROLLARY 1. Let S be a linear subspace of E and denote by $K_S$ the class of functions of J into S obtained by the orthogonal projection of elements of K into S. Then $K_S$ satisfies (i), (ii) and (iii).

Proof. Conditions (ii) and (iii) obviously hold for $K_S$, while condition (i) follows from Lemma 3.

COROLLARY 2. There is an integrable $m: J \to R$ such that $|u(t)| \leq m(t)$ a.e. in J for each $u \in K$.

Proof. By Corollary 1 $K_i=\{u_i:(u_1,\ldots,u_i,v,\ldots,u_n)\in K\}$ satisfies (i) (ii) and (iii) for each $i=1,\ldots,n$. By (ii) $\alpha_i = \sup_{v\in K_i} I(v) < +\infty$. Let $\{v^i\} \subset K_i$ be such that $I(v^i) \to \alpha_i$ as $j \to \infty$. By Lemma 2 without any loss of generality we may assume that $\{v^j\}$ is non-decreasing. Thus there exists $\lim_j v^j = \psi_i$ and by (i) $\psi_i \in K_i$. Therefore $I(v^j) \leq I(\psi_i) \leq \alpha$ and as a consequence $I(\psi_i) = \alpha$. Now for any $v \in K_i$, $I(\sup\{v,\psi_i\}) = \alpha$ and Proposition 2 implies that $u \leq \psi_i$ for each $v \in K_i$. Similarly one can prove that there is $\varphi_i \in K_i$ such that $\varphi_i \leq v$ for each $v \in K$. Since i is arbitrary we get Corollary 2 by putting $m(t)=\max(|\psi(t)|, |\varphi(t)|)$, where $\psi = (\psi_1,\ldots,\psi_n)$ and $\varphi = (\varphi_1,\ldots,\varphi_n)$.

Now we will prove the main lemma.

LEMMA 4. Suppose $\{u^i\} \subset K$ and assume that $I(u^i) \to p$ as $i \to \infty$. Then there is $v \in K$ such that

$$p \leq I(v). \tag{11}$$

Proof. Suppose that $u_j^i$ converges in the $L_1$ norm for $j=1,\ldots,k-1$ to $u_j^o$ but does not converge if $j=k$. Such a k exists, since k may be equal 1. It follows that

$$I(u_j^i) \to I(u_j^o) = p_j \quad \text{if} \quad j = 1,\ldots,k-1 \tag{12}$$

If $k-1 = n$ then (12) completes the proof of (11). If $k \leq n$ then for $j=k$ there is an $\varepsilon_o > 0$ such that for each $i_o$ there are $s \geq i_o$ and $r \geq i_o$ with $I(|u_k^s - u_k^r|) \geq \varepsilon_o$. Without any loss of generality we may assume that $u_j^i \to u_j^o$ a.e. in J as $i \to \infty$. Let us choose $i_o$ such that $|I(u_k^i) - p_k| < \varepsilon_o/4$ if $i \geq i_o$, where $p_k$ is k-th coordinate of p. By these inequalities $I(\sup(u_k^s, u_n^r)) - p_k > \varepsilon_o/4$. Put $u_k^o(t) = \lim_i \sup u_k^i(t)$ and $u^i = \sup_{m \geq i}\{u_k^m\}$. Then we see that $v^i$ is non-increasing, $v^i \to u_k^o$ as $i \to \infty$ and by the last inequality $I(v^i) \geq p_k + \varepsilon_o/4$ if $i \geq i_o$. Since by Corollary 2 the $v^i$ are bounded by an integrable function, Lebesgue theorem implies that

$$I(v^i) \to I(u_n^o) \geq p_k + \varepsilon_o/4 > 0 \tag{13}$$

It follows from Lemma 3 that there is $v \in (v_1, \ldots, v_n) \in K$ such that $v_j = u_j^o$ if $j = 1, \ldots, k$ and for this $v$ (12) and (13) imply (11) which was to be proved.

PRINCIPAL RESULTS. Again we assume throughout this section that K satisfies conditions (i),(ii) and (iii). By $e(K,\xi)$ we denote the maximal element of K with respect to "$\leq_\xi$", $\xi \in \Xi$. By Lemma 2 if $e(K,\xi)$ exists then it is uniquely defined up to a set of measure zero. We will call $e(K,\xi)$ an extremal element of K. The set of extremal elements of K will be denoted by $E(K)$, then $E(K) = \{e(K,\xi) : \xi \in \Xi\}$.

THEOREM 1. For each $\xi \in \Xi$ there exists an extremal element $e(K,\xi)$ of K corresponding to $\xi$ and

$$I(e(K,\xi)) = e(\overline{I(K)},\xi) = e(I(K),\xi) \text{ for each } \xi \in \Xi. \tag{14}$$

Proof. By (ii) the set $I(K)$ is bounded; thus, the closure $\overline{I(K)}$ of $I(K)$ is a compact subset of $E^n$. Let us fix $\xi \in \Xi$ and let $p = e(\overline{I(K)},\xi)$. By Lemma 4 there is $v \in K$ such that $p \leq_\xi I(v)$. But $I(v) \in I(K)$ implies by the definition of p that $I(v) \leq_\xi p$. Hence $I(v) = p$ and $p \in I(K)$. Let now $u \in K$ be arbitrary and $w = \operatorname{lex}_\xi \sup\{u,v\}$. We have $u \leq_\xi w$, $v \leq_\xi w$ and $p = I(v) \leq_\xi I(w) \leq_\xi p$. Therefore by Proposition 2, $v = w$. Hence $u \leq_\xi v$ for each $u \in K$. This means $v = e(K,\xi)$ and (14) is manifestly satisfied.

THEOREM 2. The set $D = \{x = I(e(K,\xi)) : \xi \in \Xi\} = I(E(K))$ is the profile $\ddot{B}$ of the convex hull B of $\overline{I(K)}$.

Proof. By (14), $D = \{x : x = e(\overline{I(K)},\xi), \xi \in \Xi\}$, and Proposition from such on a set of measure zero). Also if we knew that the number of discontinuities of $e(K,\xi)$ is finite and bounded for $\xi \in \Xi$, then by Theorem 4 there is a subset $K_*$ of $K$ composed of piecewise continuous and piecewise extremal functions such that $I(K_*) = I(K)$ and the number of discontinuities of $u$ is finite and bounded if $u \in K_*$. This is the case if $A$ in (S) is constant and $f(t,u) = B(t)u$, where the entries of $B$ are piecewise analytical and $U$ is a compact polyhedron (cf. (1),(3)). This is also the case when $G$ is a continuous set-valued function in the sense of Hausdorff with values being strictly convex and compact subsets of $E$, since in this case $e(G(t),\xi)$ is continuous in $t$ for each $\xi \in \Xi$. Note that because of strict convexity of $G(t)$, $e(G(t),\xi)$ is uniquely determined by the first vector of $\xi$.

Theorem 5, under essentially the same assumptions has been obtained by Neustadt (4). Note that as in (4) we did not make any convexity assumption concerning $K$.

Theorem 6 has some implications concerning the uniqueness of time optimal solutions of the system (S). For details, we refer the reader to (5).

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4 implies Theorem 2.

Notice that both Theorems 1 and 2 hold if $J$ is replaced by $[0,t]$, $0 < t \leq 1$ and $I$ by $I_t$. Thus if we denote by $B(t)$ the convex hull of $\overline{I_t(K)}$, then by Theorem 2 we have the equality $\ddot{B}(t) = I_t(E(K))$.

THEOREM 3. The set valued function on $J$ taking $t \to B(t)$, the convex hull of $I_t(K)$, is continuous in the Hausdorff sense; that is

$$\max_{a \in B(t), b \in B(s)} (r(a,B(s)), r(b,B(t))) \to 0 \quad \text{as} \quad |s-t| \to 0 \quad (15)$$

where $r(\ ,\ )$ stands for the distance of a point from a set in $E$.

Proof. Let $B, C$ be two compact convex subsets of $E^n$. There are $b \in B$ and $c \in C$ such that $|b-c| = r(c,B) = \max_{x \in C} r(x,B)$. Note that if $C$ were an interval, then $c$ can be assumed to be one of the ends of $C$. This remark shows that in the general case $c$ can be assumed to be an extreme point of $C$ and that there is a $\xi \in \Xi$ such that $c = e(C,\xi)$. But obviously, $r(c,B) \leq |x-c|$ for each $x \in B$. In particular, we have the inequality $|b-c| \leq |e(B,\xi) - e(C,\xi)|$ for a $\xi \in \Xi$. Therefore the distance in (18) can be estimated by $|e(B(t),\xi) - e(B(s),\xi)| \leq \int_t^s |e(K,\xi)(t)|\,dt$ for the same $\xi \in \Xi$ and Corollary 2 completes the proof.

THEOREM 4 For each $b \in B$, the convex hull of $\overline{I(K)}$, there are two sequences $\xi^1, \ldots, \xi^k \in \Xi$ and $0 = t_o < t_1 < \ldots < t_k = 1$ such that if we put

$$u(t) = e(K,\xi^i)(t) \quad \text{for} \quad t_{i-1} \leq t < t_i,\ i=1,\ldots,k, \quad (16)$$

then $k \leq n+1$ and

$$b = I(u). \quad (17)$$

Proof. The proof will be by induction with respect to $n$. Thus suppose first that $n=1$. In this case $\Xi$ consists of two elements and by Theorem 1 so does $E(K)$. That is, there are $\varphi, \psi \in K$ such that $\varphi \leq u \leq \psi$ for each $u \in K$. The set $B$ is the interval $[I(\varphi), I(\psi)]$. Consider the function

$$\lambda(t) = \int_o^t \psi(t)dt + \int_t^1 \varphi(t)dt . \quad (18)$$

Manifestly $\lambda$ is continuous and maps $J$ onto $[I(\varphi), I(\psi)]$

Thus for each $b \in B$, there is a $t_1 \in J$ such that $\lambda(t_1) = b$. Setting $u(t) = \psi(t)$ if $0 \leq t < t_1$ and $u(t) = \varphi(t)$ if $t_1 \leq t \leq 1$ we see that $u$ is of the form (16) and (17) holds.

Suppose now that $n$ is arbitrary and assume that Theorem 4 holds for $n-1$. Let $b \in B$ and take an arbitrary $\overline{\xi} \in \Xi$. Consider the function

$$x(t) = b - \int_t^1 e(K, \overline{\xi})(\tau)d\tau \tag{19}$$

Since both $x(t)$ and $B(t)$ are continuous, there is a $T \in J$ such that $x(T)$ belongs to the boundary of $B(T)$ and if $T < 1$ then $x(t) \in \inf B(t)$ for $T < t \leq 1$. Since $B(T)$ is convex and compact there is an $a \in E^n$, $|a| = 1$ such that

$$\alpha = (x(T), a) = \max\,(x, a) \quad \text{for} \quad x \in B(T). \tag{20}$$

Let $\Xi_a = \{\xi \in \Xi:\ \xi = (x^1, \ldots, x^n), x^1 = a\}$. Put $B_a = B(T) \cap \{x:\ (x,a) = \alpha\}$ and $A = I_T(K) \cap \{x:\ (x,a) = \alpha\}$. It is easy to see that $B_a$ is compact and convex, the profile $\ddot{B}_a$ of $B_a$ is equal to $\{I(e(K,\xi):\ \xi \in \Xi_a\} \subset A$. Thus $A$ is not empty and $B_a$ is equal to the convex hull of $\overline{A}$ as well as of $A$.

It follows from Proposition 3 that $I_T(u) \in A$, where $u \in K$, if and only if $(u(t), a) = \psi(t)$ a.e. in $[0,T]$, where $\psi$ has the property that for each $u \in K$ $(u(t),a) \leq \psi(t)$ a.e. in $J$. Therefore $A$ can be considered as $I_T(K_a)$ where $K_a = \{u \in K: (u(t),a) = \psi(t)$ a.e. in $J\}$ and $\psi$ is uniquely defined by $K$ and $a$. Since each $u \in K_a$ can be uniquely decomposed into the sum $v + a\psi$, where $v$ is a function of $J$ into $E_1$ and $E_1$ is the n-1 dimensional subspace perpendicular to $a$, the set $K_a$ can be considered as a class of functions of $J$ into n-1 dimensional Euclidean space. Obviously, $K_a$ satisfies conditions (i), (ii) and (iii) and by our assumption we can apply Theorem 4 to $K_a$. Hence there is a $u \in K_a$ such that $u(t) = e(K_a, \xi^i) = e(K, \xi^i)(t)$ if $t_{i-1} \leq t < t_i$, $\xi^i \in \Xi_a$, $i = 1, \ldots, k-1$, $t_o = 0 < t_1 < \ldots < t_{k-1} = T_V$ and such that

$$I_T(u) = x(T) \tag{21}$$

Setting $u(t) = e(K, \overline{\xi})(t)$ if $t_{k-1} = T \leq t \leq 1 = t_k$ (thus putting $\xi^k = \overline{\xi}$) we see that $u$ is of the form (16) and (19) and (21) implies (17). Manifestly $k \leq n+1$ since

$k-1 \leq n$.

CONCLUDING RESULTS. In this section we state three immediate consequences of the preceding theorems.

THEOREM 5. If $K$ satisfies (i),(ii) and (iii) then $I(K)$ is convex and compact.

Proof. By (iii) any function of the form (16) belongs to $K$; thus Theorem 5 follows from Theorems 2 and 4.

If $u \in K$ is an extremal element of $I$ (or $I(u)$ is an extreme point of $I(K)$) then the following implication holds (compare Proposition 1 and Theorem 2):

$$\text{if } v \in K \text{ and } I(v) = I(u) \text{ then } v = u \tag{22}$$

On the other hand one can see from the proof of Theorem 4 that if $b \in I(K)$ is not an extreme point of $I(K)$ then there are at least two different $u, v \in K$ such that $I(u) = I(v) = b$. Therefore we have

THEOREM 6. If $K$ satisfies (i),(ii) and (iii) and $u \in K$ then $u$ is an extremal element of $K$ if and only if the implication (22) holds for $u$.

Let $K_0$ denote the class we obtained by closing $E(K)$ with respect to property (iii). Elements of $K_0$ may be referred to as piecewise extremal elements of $K$.

THEOREM 7. If $K_1 \subset K$ satisfies (iii) and $I(K_1) = I(K)$, then $K_0 \subset K_1$

Proof. By Theorem 6, $K_1$ must contain $E(K)$. The definition of $K_0$ and $K_1$ satisfing (iii) imply $K_1 \supset K_0$.

Theorem 7 says that $K_0$ is the smallest subclass of $K$ satisfying (iii) and having the same range of integrals as $K$.

Let us observe that if $K = \{u \in M : u(t) \in G(t) \text{ a.e. in } J\}$ and $G$ is a measurable set-valued function with values being compact subsets of $E$ then $e(K,\xi)(t) = e(G(t),\xi)$ (cf.(6)). So in that case the extremal elements of $K$ can be computed if one knows $G$.

Under some more restrictive assumptions LaSalle (2), Halkin (1) and Levinson (3) proved that the "bang-bang" controls (elements of $K_0$ in our case) can be chosen to be piecewise constant or piecewise continuous. From Theorem 7 it follows that this can be the case if and only if each extremal element of $K$ is piecewise continuous (or differs